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ON THE STRUCTURE OF ORTHOMODULAR POSETS

P.J. GUSTAV MEYER

Department of Mathematics, Monash University, Melbourne, Australia

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Abstract. It is shown how all orthomodular posets (of various kinds) are constructible from families of sets satisfying various conditions, usually with the generating family emerging as identical with (or contained in) the family of frames (that is, maximal orthogonal subsets of the non-zero elements) of the constructed orthomodular poset.

1. Introduction

In his investigations into orthomodular posets (hereinafter abbreviated "o.m.p.") and quantum logic, Finch [1, 2] has considered the construction of o.m.p. from sets of "overlapping" Boolean lattices having a common 0-element and a common 1-element, and satisfying five other conditions. Such a collection of Boolean lattices Finch terms a *logical structure*, and he has shown in [1] that from any given logical structure there can be constructed an o.m.p., and that all o.m.p. can be so constructed. This paper extends this approach significantly by considering what conditions imposable on an arbitrary family of *sets* are sufficient for the construction of an o.m.p. therefrom, and in such a way that all o.m.p. can be so constructed from some such family of sets. Various sets of conditions are obtained, depending on whether the resulting o.m.p. is required to be completely orthomodular, atomic, etc.

Elements x and y in an orthocomplemented poset $(P, \leq, \perp)$ are *orthogonal* if $x \leq y^{\perp}$, and P is an *orthoposet* if the least upper bound $x \vee y$ exists in P for any orthogonal $x, y \in P$. $X \subseteq P$ is *orthogonal* if all pairs of distinct elements of X are orthogonal. X is a *maximal orthogonal subset* of $Y \subseteq P$ if X is an orthogonal subset of Y and there is no $y \in Y \setminus X$ such that $X \cup \{y\}$ is orthogonal. $F \subseteq P$ is a *frame* of P if F is a maximal orthogonal subset of $P \setminus \{0\}$, where 0 is the least element of P. We follow

Finch [1] in using the term "frame" in this sense. A frame F is *complete* if $\bigvee X$ exists in P for any $X \subseteq F$. For a complete frame F of P, define $B(F) = \{\bigvee X\colon X \subseteq F\}$.

An orthoposet P is *orthomodular* if $x = y^{\perp}$ for any orthogonal $x, y \in P$ such that $x \vee y = 1$. An o.m.p. is *completely* orthomodular if it is a complete orthoposet, that is, if $\bigvee X$ exists in P for any orthogonal $X \subseteq P$. x *dominates* y if $y \leq x$, and $Y \subseteq P$ is a *section* of P if every non-zero element of P dominates some element of Y.

We state the following elementary lemmas for later reference.

Lemma 1.1. *Let $(P, \leq, \perp)$ be an orthocomplemented poset and let $S \subseteq P \setminus \{0\}$, then S is a frame of $P \Leftrightarrow S$ is orthogonal and $\bigvee S = 1$.*

Lemma 1.2. *Let A be a section of the o.m.p. $(P, \leq, \perp)$. Let $y \in P$ and define $Y = \{x \in A\colon x \leq y\}$, then*

(i) *Y contains a maximal orthogonal subset $[y]$, and*

(ii) *if $\bigvee[y]$ exists in P, then $\bigvee[y] = y$.*

Lemma 1.3. *An orthoposet $(P, \leq, \perp)$ is orthomodular $\Leftrightarrow$ for any $x, y \in P$, if $x \leq y$, then there is a complete frame F of P such that $x, y \in B(F)$.*

2. Consistent complete families

Let $\mathbf{N}$ and E denote respectively the set of natural numbers and the set of even natural numbers, and let $|X|$, $\mathcal{P}(X)$ and $\emptyset$ denote respectively the number of elements in X, the power set of X, and the empty set

A *family* is any nonempty set of nonempty sets. For any family $\mathcal{F} = \{F_\beta\colon \beta \in B\}$, F^* will denote the set of all subsets of elements of $\mathcal{F}$, that is, $F^* = \bigcup\{\mathcal{P}(F_\beta)\colon \beta \in B\}$. We will be concerned, basically, with multiply applied operations on elements on F^*, for which we will need a special notation, as follows.

Let $\mathcal{F} = \{F_\beta\colon \beta \in B\}$ be a family, $X \in F^*$, and $\beta \in B$. Define $X[\beta]$ as $F_\beta \setminus X$ if $X \subseteq F_\beta$ (otherwise undefined). Let $n \in \mathbf{N}$ and $(\beta_i) \in B^n$, then define

$$X[\beta_1, \beta_2, ..., \beta_n] = \begin{cases} X & \text{if } n = 0\,, \\ X[\beta_1, ..., \beta_{n-1}]\,[\beta_n] & \text{otherwise}\,. \end{cases}$$

Then

$$X[\beta_1, ..., \beta_n] = F_{\beta_n} \setminus (F_{\beta_{n-1}} \setminus (F_{\beta_{n-2}} \setminus ... \setminus (F_{\beta_1} \setminus X) ...))$$

provided that for all $i \in \mathbf{N}$ such that $1 \leq i \leq n$,

$$F_{\beta_{i-1}} \setminus (F_{\beta_{i-2}} \setminus ... \setminus (F_{\beta_1} \setminus X) ...) \subseteq F_{\beta_i} .$$

If any of these n inclusions does not hold, then $X[\beta_1, \beta_2, ..., \beta_n]$ is undefined.

We adopt the following abbreviations: $X[\beta_1, ..., \beta_n]$ is abbreviated to $X[\beta_{1\to n}]$, $X[\beta_n, ..., \beta_1]$ to $X[\beta_{1\leftarrow n}]$, and $X[\beta_1, ..., \beta_n, \gamma_1, ..., \gamma_m]$ to $X[\beta_{1\to n}, \gamma_{1\to m}]$. Other abbreviations, such as $X[\beta, \gamma_{1\to n}, \lambda]$ should be clear from this account.

Intuitively, the term $X[\beta_{1\to n}]$ (for $n > 0$) should be thought of as the result, beginning with $X \in F^*$, of successively taking complements in $F_{\beta_1}, F_{\beta_2}, ..., F_{\beta_n}$, provided that each element of F^* occurring in this process is contained in the next F_β. Some families may be so poorly connected that this process cannot be carried very far (without redundancy), whereas sufficiently rich families may allow the process to continue indefinitely (without redundancy).

An occurrence of a term such as $X[\beta_{1\to m}]$ in an equation is to be understood as being accompanied by the tacit assertion that it is defined (otherwise the equation is meaningless). If such a term is to be introduced in the course of a proof, it must (strictly speaking) be shown that it is defined, although usually this would be clear. We will sometimes indicate that the "n" in $X[\beta_{1\to n}]$ denotes an even natural number by writing "($n \in E$)" shortly thereafter; in other cases $n \in \mathbf{N}$. Whenever $\mathcal{F}$ is said to be a family, we will mean $\mathcal{F} = \{F_\beta : \beta \in B\}$ for some index set B.

Lemma 2.1. *Let $\mathcal{F}$ be a family, and let X, $Y \in F^*$, then for any $n \in \mathbf{N}$, if $X[\beta_{1\to n}] = Y$, then $Y[\beta_{1\leftarrow n}] = X$.*

Proof. By induction on n.

Note that $X[\beta_{1\to n}] \subseteq Y$ does not imply $Y[\beta_{1\leftarrow n}] \subseteq X$ since Y need not be contained in F_{β_n}.

For any family $\mathcal{F}$, let $\equiv$ denote the binary relation on F^* defined as follows: For $X, Y \in F^*$, $X \equiv Y \Leftrightarrow$ for some $n \in E$, there is $(\beta_i) \in B^n$ such that $X[\beta_{1\to n}] = Y$.

Lemma 2.2. *$\equiv$ is an equivalence relation on F^*.*

Proof. Reflexivity and transitivity are elementary. Symmetry follows from the previous lemma.

A family $\mathcal{F}$ is *consistent* (respectively *complete*) if for any $Y \in F^*$ such that $Y[\beta_{1\to m}] \subseteq F_\gamma$ ($m \in E$), $Y \cap (Y[\beta_{1\to m}, \gamma]) = \emptyset$ (respectively $Y \cup (Y[\beta_{1\to m}, \gamma]) \in \mathcal{F}$).

Lemma 2.3. *Let $\mathcal{F}$ be a consistent family, then*
(i) *if $F_\lambda \subseteq F_\gamma \cup F_\mu$, then*

$$(F_\gamma \setminus F_\lambda) \cap (F_\mu \setminus (F_\lambda \setminus F_\gamma)) = \emptyset$$

and $F_\gamma \cap F_\mu \subseteq F_\lambda$;
(ii) *if $F_\lambda \subseteq F_\gamma$, then $F_\lambda = F_\gamma$.*

Proof. (i) Suppose $F_\lambda \subseteq F_\gamma \cup F_\mu$, then $F_\lambda \setminus F_\gamma \subseteq F_\mu$. $(F_\gamma \setminus F_\lambda)[\gamma] = F_\gamma \cap F_\lambda$, so $(F_\gamma \setminus F_\lambda)[\gamma, \lambda] = F_\lambda \setminus F_\gamma \subseteq F_\mu$. $\mathcal{F}$ is consistent so

$$(F_\gamma \setminus F_\lambda) \cap (F_\gamma \setminus F_\lambda)[\gamma, \lambda, \mu] = \emptyset .$$

Thus $(F_\gamma \setminus F_\lambda) \cap (F_\mu \setminus (F_\lambda \setminus F_\gamma)) = \emptyset$, from which it follows that $(F_\mu \cap F_\gamma) \setminus F_\lambda = \emptyset$, so $F_\gamma \cap F_\mu \subseteq F_\lambda$.
(ii) Suppose $F_\lambda \subseteq F_\gamma$, then $F_\lambda \subseteq F_\gamma \cup F_\gamma$, so by (i), $F_\gamma \subseteq F_\lambda$.

Lemma 2.4. *Let $\mathcal{F}$ be a consistent family, then for any $X \in F^*$, if $X[\gamma_{1\to n}], X[\beta_{1\to m}] \subseteq F_\mu$ ($m, n \in E$), then $X[\gamma_{1\to n}] = X[\beta_{1\to m}]$.*

Proof. Let $X[\gamma_{1\to n}], X[\beta_{1\to m}] \subseteq F_\mu$ ($m, n \in E$). Define $Y = X[\gamma_{1\to n}]$, then

$$Y[\gamma_{1\leftarrow n}, \beta_{1\to m}] = X[\beta_{1\to m}] \subseteq F_\mu .$$

$m + n \in E$, so by the consistency of $\mathcal{F}$,

$$Y \cap Y[\gamma_{1\leftarrow n}, \beta_{1\rightarrow m}, \mu] = \emptyset ,$$

from which it follows that $X[\gamma_{1\rightarrow n}] \subseteq X[\beta_{1\rightarrow m}]$. The converse inclusion is proved similarly.

Corollary 2.5. *Let $\mathcal{F}$ be a consistent family and let $W, Z \in F^*$ such that $Z \subseteq W \equiv Z$, then $Z = W$.*

Lemma 2.6. *Let $\mathcal{F}$ be a complete family, then if $F_\lambda \subseteq F_\gamma \cup F_\mu$, then there is $\beta \in B$ such that*

$$F_\beta = (F_\gamma \setminus F_\lambda) \cup (F_\mu \setminus (F_\lambda \setminus F_\gamma)) .$$

Proof. Define $Y = F_\gamma \setminus F_\lambda$, then $Y[\gamma, \lambda] = F_\lambda \setminus F_\gamma \subseteq F_\mu$. $\mathcal{F}$ is complete so $Y \cup Y[\gamma, \lambda, \mu] \in \mathcal{F}$. $Y[\gamma, \lambda, \mu] = F_\mu \setminus (F_\lambda \setminus F_\gamma)$, and the lemma follows.

Lemma 2.7. *Let $\mathcal{F}$ be a complete family, then for any $X, Y, Z \in F^*$ such that $X \subseteq Y$ and $Y[\mu_{1\rightarrow m}] \subseteq Z \subseteq F_\gamma$ $(m \in E)$, there is $\beta \in B$ such that $X \cup (F_\gamma \setminus Z) \subseteq F_\beta$.*

Proof. $F_\gamma \setminus Z \subseteq Y[\mu_{1\rightarrow m}, \gamma]$, so

$$X \cup (F_\gamma \setminus Z) \subseteq Y \cup Y[\mu_{1\rightarrow m}, \gamma] .$$

By the completeness of $\mathcal{F}$ there is $\beta \in B$ such that $F_\beta = Y \cup Y[\mu_{1\rightarrow m}, \gamma]$, so $X \cup (F_\gamma \setminus Z) \subseteq F_\beta$.

Lemma 2.8. *Let $\mathcal{F}$ be a consistent complete family, then for any $X, X', Z' \in F^*$, if $X \equiv X' \subseteq Z'$, then there is $\beta \in B$ and $Z \subseteq F_\beta$ such that $X \subseteq Z \equiv Z'$.*

Proof. Let $X \equiv X' \subseteq Z' \subseteq F_\gamma$, then $X' = X[\beta_{1\rightarrow n}]$ for some $(\beta_i) \in B^n$ $(n \in E)$. By Lemma 2.7 there is $\beta \in B$ such that $X \cup (F_\gamma \setminus Z') \subseteq F_\beta$. $\mathcal{F}$ is consistent and $X[\beta_{1\to n}] \subseteq F_\gamma$, so $X \cap X[\beta_{1\to n}, \gamma] = \emptyset$. Now $F_\gamma \setminus Z' \subseteq F_\gamma \setminus X' = X[\beta_{1\to n}, \gamma]$, so $X \cap (F_\gamma \setminus Z') = \emptyset$. Thus $X \subseteq F_\beta \setminus (F_\gamma \setminus Z') = Z'[\gamma, \beta]$. Define $Z = Z'[\gamma, \beta]$, then $Z \subseteq F_\beta$ and $X \subseteq Z \equiv Z'$.

The "dual" of Lemma 2.8 is: "For any consistent complete family $\mathcal{F}$ and for any $X, Z, Z' \in F^*$, if $X \subseteq Z \equiv Z'$, then there is $X' \in F^*$ such that $X \equiv X' \subseteq Z'$." This dual is false, as shown by the following counter-example. Let

$$\mathcal{F} = \{\{a, b, c\}, \{c, d, e\}\}$$

and let $X = \{a\}$, $Z = \{a, b\}$ and $Z' = \{d, e\}$.

The following three lemmas are not difficult to prove, but for the sake of brevity we omit the proofs. In the proof of Lemma 2.9, (ii) follows from (i); Lemma 2.10 is proved using Lemmas 2.3(i) and 2.9(ii); and Lemma 2.11 follows from Lemmas 2.2, 2.8 and Corollary 2.5.

Lemma 2.9. *Let $\mathcal{F}$ be a consistent complete family, then*

(i) *for any $X \in F^*$, if $X[\mu_{1\to m}]$ is defined for $m \in E \setminus \{0\}$, then there is $\beta \in B$ such that $X[\mu_{1\to m}, \beta, \mu_1] = X$;*

(ii) *for any $X, Y \in F^*$, if $X \equiv Y$, then there are $\gamma, \lambda \in B$ such that $X = Y[\gamma, \lambda]$.*

Lemma 2.10. *Let $\mathcal{F}$ be a consistent complete family and let $X, Y \in F^*$ such that $X \subseteq Y$ and $X, Y[\beta_{1\to m}] \subseteq F_\alpha$ $(m \in E)$, then $X \subseteq Y[\beta_{1\to m}]$.*

Lemma 2.11. *Let $\mathcal{F}$ be a consistent complete family and let $Z_1, Z_2, W_1, W_2, W_3, W_4 \in F^*$ such that*

$$Z_1 \equiv W_1 \subseteq W_2 \equiv Z_2\,, \qquad Z_2 \equiv W_3 \subseteq W_4 \equiv Z_1\,,$$

then $Z_1 \equiv Z_2$.

3. Orthocomplemented posets

For any family $\mathcal{F}$, let $L(\mathcal{F})$ denote the set of all equivalence classes of F^* with respect to $\equiv$ (recall Lemma 2.2). These equivalence classes will eventually become the elements of the various o.m.p. constructed from families with various sets of conditions imposed on them. Let $C: F^* \to L(\mathcal{F})$ be such that $C(X)$ is the equivalence class containing $X \in F^*$. Clearly, for any $x \in L(\mathcal{F})$, $x \neq \emptyset$, and $\emptyset \in x \Leftrightarrow x = C(\emptyset)$.

Lemma 3.1. *Let $\mathcal{F}$ be a consistent family, then for any $\beta \in B$, the restriction of C to $\mathcal{P}(F_\beta)$ is an injection (is one-one).*

Proof. By Lemma 2.4.

For any consistent family $\mathcal{F}$ and any $\beta \in B$, define $B_\beta = \{C(X)\colon X \subseteq F_\beta\}$ (that is, $C(\mathcal{P}(F_\beta))$), then $B_\beta \subseteq L(\mathcal{F})$. For any $x \in B_\beta$ there is (by Lemma 3.1) a unique $X \in \mathcal{P}(F_\beta)$ such that $C(X) = x$, which X we denote by x_β. If $x \in B_\beta \cap B_\gamma$, then $C(x_\beta) = x = C(x_\gamma)$, so $x_\beta \equiv x_\gamma$.

We define a binary relation $\leq_\beta$ on B_β, and a function $N_\beta : B_\beta \to B_\beta$ thus: For any $x, y \in B_\beta$, let $x \leq_\beta y \Leftrightarrow x_\beta \subseteq y_\beta$. For $x \in B_\beta$, define $N_\beta x = C(F_\beta \setminus x_\beta)$. Clearly, $\leq_\beta$ and N_β are well-defined, and it is also clear that $(B_\beta, \leq_\beta, N_\beta)$ is a Boolean lattice since (under the restriction of C to $\mathcal{P}(F_\beta)$) it is isomorphic to the Boolean lattice of all subsets of F_β. $\{B_\beta\colon \beta \in B\}$ is thus a set of Boolean lattices, said to be *generated by* $\mathcal{F}$, and denoted by $\mathcal{L}(\mathcal{F})$.

Adapting a procedure in [1], we define a binary relation $\leq$ on $L(\mathcal{F})$, and a function $\perp : L(\mathcal{F}) \to L(\mathcal{F})$ as follows: For $x, y \in L(\mathcal{F})$, let $x \leq y \Leftrightarrow x \leq_\beta y$ for some $\beta \in B$. For $x \in L(\mathcal{F})$, define $x^\perp = N_\beta x$ for any $\beta \in B$ such that $x \in B_\beta$. It will be shown that if $x \in B_\alpha \cap B_\beta$, then $N_\alpha x = N_\beta x$, so $\perp$ is well-defined. The structure $(L(\mathcal{F}), \leq, \perp)$ is termed *the logic generated by* $\mathcal{F}$. The construction of $(L(\mathcal{F}), \leq, \perp)$ depends on the consistency of $\mathcal{F}$ but does not require that $\mathcal{F}$ be complete.

Following Finch [1, 2], a set

$$\{(B_\beta, \leq_\beta, N_\beta)\colon \beta \in B\}$$

of Boolean lattices is a *weak logical structure* if:

(i) Each B_β has the same 0-element.

(ii) For $x, y \in B_\alpha \cap B_\beta$, $x \leq_\alpha y \Leftrightarrow x \leq_\beta y$.

(iii) If $x \leq_\alpha y$ and $y \leq_\beta z$, then there is $\gamma \in B$ such that $x \leq_\gamma z$.

(iv) If $x \in B_\alpha \cap B_\beta$, then $N_\alpha x = N_\beta x$.

A weak logical structure is a *logical structure* if in addition it satisfies:

(v) If $x, y \in B_\alpha \cap B_\beta$, then $x \vee_\alpha y = x \vee_\beta y$ (that is, the least upper bound of x and y is the same in B_α as in B_β).

(vi) If $x \leq_\alpha N_\alpha y$, $x \leq_\beta z$ and $y \leq_\gamma z$, then there is $\delta \in B$ such that $x, y, z \in B_\delta$.

Lemma 3.2. *Let $\mathcal{F}$ be a consistent complete family, then the set $\mathcal{L}(\mathcal{F})$ of Boolean lattices generated by $\mathcal{F}$ is a weak logical structure.*

Proof. We show that $\mathcal{L}(\mathcal{F})$ satisfies conditions (i)–(iv) above.

(i) Clearly, each B_β has the same 0-element, namely $C(\emptyset) = \{\emptyset\}$. (The 1-element of each B_β is $\mathcal{F}$.)

(ii) Let $x, y \in B_\alpha \cap B_\beta$ such that $x \leq_\beta y$, then $x_\alpha \equiv x_\beta \subseteq y_\beta$. By Lemma 2.8 there is $Y \in F^*$ such that $x_\alpha \subseteq Y \equiv y_\beta$. $y_\beta \equiv y_\alpha$ so $Y \equiv y_\alpha$, so for some $(\beta_i) \in B^m$ $(m \in E)$, $y_\alpha = Y[\beta_{1 \to m}]$. By Lemma 2.10, $x_\alpha \subseteq Y[\beta_{1 \to m}]$, so $x = C(x_\alpha) \leq_\alpha C(y_\alpha) = y$. Similarly, if $x \leq_\alpha y$, then $x \leq_\beta y$.

(iii) Let $x \leq_\alpha y$ and $y \leq_\beta z$, then $x_\alpha \subseteq y_\alpha$ and $y_\beta \subseteq z_\beta$. $y_\alpha \equiv y_\beta$ so by Lemma 2.8 there is $\gamma \in B$ and $Z \subseteq F_\gamma$ such that $y_\alpha \subseteq Z \equiv z_\beta$. Thus

$$x_\alpha \subseteq y_\alpha \subseteq Z \subseteq F_\gamma ,$$

so $x = C(x_\alpha) \leq_\gamma C(Z) = C(z_\beta) = z$.

(iv) Let $x \in B_\alpha \cap B_\beta$, then $x_\alpha \equiv x_\beta$, so $x_\alpha [\beta_{1 \to n}] = x_\beta$ for some $(\beta_i) \in B^n$ $(n \in E)$. Since

$$(F_\alpha \setminus x_\alpha)[\alpha, \beta_{1 \to n}, \beta] = F_\beta \setminus x_\beta ,$$

we have $F_\alpha \setminus x_\alpha \equiv F_\beta \setminus x_\beta$, and so $N_\alpha x = N_\beta x$. Thus $\mathcal{L}(\mathcal{F})$ is a weak logical structure.

Let $\mathcal{F}$ be a family, then $x \in \bigcup \mathcal{F}$ is *aloof from* F_β if $x \notin F_\beta$ and for all $y \in F_\beta$ there is $\mu \in B$ such that $x, y \in F_\mu$. An element $x \in \bigcup \mathcal{F}$ is *aloof* if x is aloof from some F_β in $\mathcal{F}$. For $x \in \bigcup \mathcal{F}$, define $x_c = C(\{x\})$. It is easily shown that if $\mathcal{F}$ is consistent, then (by Lemma 2.4) x_c is a *nonzero* element of $(L(\mathcal{F}), \leq, \perp)$. Define $(F_\beta)_c = \{y_c : y \in F_\beta\}$.

Theorem 3.3. *Let $\mathcal{F}$ be a consistent complete family, then the logic $(L(\mathcal{F}), \leq, \perp)$ generated by $\mathcal{F}$ is an orthocomplemented poset. Further, for any $\beta \in B$, $(F_\beta)_c$ is an orthogonal subset of $L(\mathcal{F}) \setminus \{0\}$, and is a frame of $L(\mathcal{F}) \Leftrightarrow$ there is no element of $\bigcup \mathcal{F}$ which is aloof from F_β.*

Proof. By Lemma 3.2, $\mathcal{L}(\mathcal{F})$ is a weak logical structure. It is shown in [2] that (in the terminology of that paper) the logic associated with a weak logical structure is an orthocomplemented poset. Since $L(\mathcal{F})$ is the logic associated with $\mathcal{L}(\mathcal{F})$, $L(\mathcal{F})$ is an orthocomplemented poset.

Let $\beta \in B$. For $x, y \in F_\beta$, if $x_c \neq y_c$, then $x \neq y$, so $\{x\} \subseteq F_\beta \setminus \{y\}$. Thus

$$x_c = C(\{x\}) \leq C(F_\beta \setminus \{y\}) = y_c^\perp ,$$

so $\{y_c : y \in F_\beta\}$ is orthogonal. Since $\mathcal{F}$ is consistent, $y_c \neq 0$ for all $y \in F_\beta$.

Suppose $x \in \bigcup\mathcal{F}$ is aloof from F_β. $\{x_c\} \cup (F_\beta)_c$ is orthogonal, so if $(F_\beta)_c$ is a frame, then $x_c \in (F_\beta)_c$. Suppose so, then $x_c = y_c$ for some $y \in F_\beta$, so $\{y\} = \{x\}[\beta_{1\to n}]$ for some $(\beta_i) \in B^n$ $(n \in E)$. Since x is aloof from F_β, $x, y \in F_\mu$ for some $\mu \in B$, so $\{x\}, \{x\}[\beta_{1\to n}] \subseteq F_\mu$. By Lemma 2.4, $\{x\} = \{x\}[\beta_{1\to n}] = \{y\}$, so $x = y$. Thus $x \in F_\beta$, a contradiction, so $(F_\beta)_c$ is not a frame of $L(\mathcal{F})$.

Conversely, suppose that $(F_\beta)_c$ is not a frame of $L(\mathcal{F})$. Then there is a nonzero $z \in L(\mathcal{F}) \setminus (F_\beta)_c$ such that $\{z\} \cup (F_\beta)_c$ is orthogonal. $z \neq 0$ so there is nonempty $X \in z$. Let $x \in X$. If $x \in F_\beta$, then $x_c = 0$, so $x \notin F_\beta$. We will show that x is aloof from F_β.

Let $y \in F_\beta$, then $x_c \leq z \leq y_c^\perp$ so $y_c \leq_\gamma x_c^\perp$ for some $\gamma \in B$. Hence for some $Y', X' \subseteq F_\gamma$,

$$\{y\} \equiv Y' \subseteq X' \equiv F_\delta \setminus \{x\}$$

for any $\delta \in B$ such that $x \in F_\delta$. By Lemma 2.8 there is $X'' \in F^*$ such that

$$\{y\} \subseteq X'' \equiv X' \equiv F_\delta \setminus \{x\} .$$

For some $(\beta_i) \in B^m$ $(m \in E)$, $F_\delta \setminus \{x\} = X''[\beta_{1\to m}]$, so $\{y\} \subseteq X''$ and $X''[\beta_{1\to m}] \subseteq F_\delta$. By Lemma 2.7 there is $\mu \in B$ such that

$$\{y\} \cup (F_\delta \setminus (X''[\beta_{1\to m}])) \subseteq F_\mu ,$$

so $\{y\} \cup \{x\} \subseteq F_\mu$. Hence for any $y \in F_\beta$ there is $\mu \in B$ such that $x, y \in F_\mu$, which completes the proof.

We prove the following lemma here for later reference (*atom* is defined in Section 8).

Lemma 3.4. *Let $\mathcal{F}$ be consistent and complete, and such that $|F_\gamma \setminus F_\lambda| \neq 1$ for all $\gamma, \lambda \in B$, then $(\bigcup \mathcal{F})_c$ is the set of atoms of $(L(\mathcal{F}), \leq, \perp)$.*

Proof. Let $x \in \bigcup \mathcal{F}$ and $y \in L(\mathcal{F})$ such that $y \leq x_c$, then there is $\beta \in B$ such that $y_\beta \subseteq X \equiv \{x\}$, so by Lemma 2.9(ii), $y_\beta \subseteq \{x\}[\gamma, \lambda]$. $F_\gamma \setminus \{x\} \subseteq F_\lambda$, so $F_\gamma \setminus F_\lambda \subseteq \{x\}$. Since $|F_\gamma \setminus F_\lambda| \neq 1$, we have (using Lemma 2.3(ii)) that $F_\gamma = F_\lambda$, so $y_\beta \subseteq \{x\}$. If $y_\beta = \emptyset$, then $y = 0$. If $y_\beta = \{x\}$, then $y = x_c$, so x_c is an atom of $L(\mathcal{F})$. Conversely, suppose z is an atom of $L(\mathcal{F})$. Since $z \neq 0$, there is $X \in z$ such that $X \neq \emptyset$. Let $x \in X$, then $x_c \leq C(X) = z$. $x_c \neq 0$, so $x_c = z$, so $z \in (\bigcup \mathcal{F})_c$.

4. Orthomodular posets

Consistency and completeness gives us an orthocomplemented poset. To ensure orthomodularity we require a third condition. A family $\mathcal{F}$ is *compact* if, whenever $Z[\beta_{1\to k}]$ is defined ($k \in E$), for any $\alpha \in B$ there is $(\delta_i) \in B^n$ ($n \in E$) such that

$$F_\alpha \cap (Z \cup Z[\beta_{1\to k}]) \subseteq Z[\delta_{1\to n}] .$$

The following lemma is a direct consequence of this definition.

Lemma 4.1. *Let $\mathcal{F}$ be a compact family, then for any $X, Y, Z', Z'' \in F^*$, if $X, Y \subseteq F_\alpha$, $X \subseteq Z'$, $Y \subseteq Z''$ and $Z' \equiv Z''$, then there is $\beta \in B$ and $Z \subseteq F_\beta$ such that $X \cup Y \subseteq Z \equiv Z'$.*

Lemma 4.2. *Let $\mathcal{F}$ be a complete compact family and let $Z \subseteq F_\beta$ such* *that* $Z[\mu_{1\to m}]$ *is defined* ($m \in E$), *then for any* $\alpha \in B$ *there is* $\nu \in B$ *such that*

$$(F_\alpha \cap (Z \cup Z[\mu_{1\to m}])) \cup Z[\beta] \subseteq F_\nu .$$

Proof. Since $\mathcal{F}$ is compact,

$$F_\alpha \cap (Z \cup Z[\mu_{1\to m}]) \subseteq Z[\delta_{1\to n}] .$$

The result follows from Lemma 2.7 by taking $X = F_\alpha \cap (Z \cup Z[\mu_{1\to m}])$ and $Y = Z[\delta_{1\to n}]$.

Lemma 4.3. *Let* $\mathcal{F}$ *be a consistent complete compact family, and let* $X, Y \subseteq F_\alpha$ *and* $X', Y' \subseteq F_\beta$ *such that* $X \equiv X'$ *and* $Y \equiv Y'$, *then* $(X \cup Y) \equiv (X' \cup Y')$.

Proof. Since $X \equiv X'$ and $Y \equiv Y'$, there are, by Lemma 2.9(ii), $\gamma_1, \gamma_2, \lambda_1, \lambda_2 \in B$ such that $X' = X[\gamma_1, \gamma_2]$ and $Y' = Y[\lambda_1, \lambda_2]$.

Since $F_{\gamma_2} \setminus X' \subseteq F_{\gamma_1}, F_{\gamma_2} \subseteq F_{\gamma_1} \cup F_\beta$, so by Lemma 2.6 there is $\mu_1 \in B$ such that

$$F_{\mu_1} = (F_{\gamma_1} \setminus F_{\gamma_2}) \cup (F_\beta \setminus (F_{\gamma_2} \setminus F_{\gamma_1})) .$$

By Lemma 4.2 there is $\mu_2 \in B$ such that

$$(F_{\mu_1} \cap (Y \cup Y')) \cup Y[\alpha] \subseteq F_{\mu_2} .$$

Define Y_0 as the left-hand expression, then since $Y_0 \subseteq F_{\mu_2}$, we have

$$\begin{aligned} Y_0 \setminus ((X \cup Y)[\alpha]) &\subseteq (Y_0 \cup Y_0[\mu_2]) \setminus ((X \cup Y)[\alpha]) \\ &= F_{\mu_2} \setminus (X \cup Y)[\alpha] \\ &= (X \cup Y)[\alpha, \mu_2] \end{aligned}$$

since $(X \cup Y)[\alpha] \subseteq Y[\alpha] \subseteq F_{\mu_2}$.

$$F_{\gamma_2} = X' \cup (F_{\gamma_2} \setminus X') \subseteq X' \cup F_{\gamma_1} ,$$

so $F_{\gamma_2} \setminus F_{\gamma_1} \subseteq X' \subseteq X' \cup Y'$. Thus $(X' \cup Y')[\beta] \subseteq F_{\mu_1}$, so $(X' \cup Y')$ $[\beta, \mu_1]$ is defined.

$$F_{\lambda_1} = Y \cup (F_{\lambda_1} \setminus Y) \subseteq F_\alpha \cup F_{\lambda_2} ,$$

so by Lemma 2.3(i), $F_\alpha \cap F_{\lambda_2} \subseteq F_{\lambda_1}$. Now $F_\alpha \cap Y' = (F_\alpha \cap F_{\lambda_2} \setminus F_{\lambda_1}) \cup (Y \cap F_{\lambda_2})$, so $F_\alpha \cap Y' \subseteq Y$. Thus $(F_{\mu_1} \cap Y') \cap F_\alpha \subseteq Y$, and since $(F_{\mu_1} \cap Y') \setminus F_\alpha \subseteq Y' \setminus F_\alpha$, we have

$$F_{\mu_1} \cap Y' \subseteq Y \cup (Y' \setminus F_\alpha) .$$

Now $F_{\mu_1} \cap X' \subseteq X$, so

$$F_{\mu_1} \cap (X' \cup Y') \subseteq (X \cup Y) \cup (Y' \setminus F_\alpha) .$$

$F_{\mu_1} \setminus F_\beta = (F_{\gamma_1} \setminus F_{\gamma_2}) \setminus F_\beta$ and $F_{\gamma_1} \setminus F_{\gamma_2} \subseteq X$, so $F_{\mu_1} \setminus F_\beta \subseteq X$. Thus

$$\begin{aligned}(X' \cup Y')[\beta, \mu_1] &= (F_{\mu_1} \setminus F_\beta) \cup (F_{\mu_1} \cap (X' \cup Y')) \\ &\subseteq (X \cup Y) \cup (Y' \setminus F_\alpha) ,\end{aligned}$$

so

$$\begin{aligned}(X' \cup Y')[\beta, \mu_1] &\subseteq F_{\mu_1} \cap ((X \cup Y) \cup (Y' \setminus F_\alpha)) \\ &= Y_0 \setminus ((X \cup Y)[\alpha]) ,\end{aligned}$$

where Y_0 was defined above.

It was proved above that

$$Y_0 \setminus ((X \cup Y)[\alpha]) \subseteq (X \cup Y)[\alpha, \mu_2] ,$$

so $(X' \cup Y')[\beta, \mu_1] \subseteq (X \cup Y)[\alpha, \mu_2]$. By interchanging α and β, X and Y', and Y and X', and substituting ν_1 and ν_2 for μ_1 and μ_2 respectively, an exactly parallel argument gives us

$$(Y \cup X)[\alpha, \nu_1] \subseteq (Y' \cup X')[\beta, \nu_2] .$$

The result now follows from these two inclusions by Lemma 2.11.

Lemma 4.4. *Let $\mathcal{F}$ be a consistent complete compact family, then the set $\mathcal{L}(\mathcal{F})$ of Boolean lattices generated by $\mathcal{F}$ is a logical structure.*

Proof. By Lemma 3.2, $\mathcal{L}(\mathcal{F})$ is a weak logical structure, so we have only to show that it satisfies the conditions (v) and (vi) given just prior to that lemma.

(v) Let $x, y \in B_\alpha \cap B_\beta$. Clearly, $x \vee_\alpha y = C(x_\alpha \cup y_\alpha)$ and $x \vee_\beta y = C(x_\beta \cup y_\beta)$. $x_\alpha \equiv x_\beta$ and $y_\alpha \equiv y_\beta$, so by Lemma 4.3, $(x_\alpha \cup y_\alpha) \equiv (x_\beta \cup y_\beta)$. Thus $x \vee_\alpha y = x \vee_\beta y$.

(vi) Let $x, y \in B_\alpha$, $x \leq_\beta z$ and $y \leq_\gamma z$. $x_\alpha \equiv x_\beta$ so by Lemma 2.8 there is $Z' \in F^*$ such that $x_\alpha \subseteq Z' \equiv z_\beta$. Similarly there is $Z'' \in F^*$ such that $y_\alpha \subseteq Z'' \equiv z_\gamma$. $Z' \equiv Z''$, so by Lemma 4.1 there is $\nu \in B$ and $Z \subseteq F_\nu$ such that

$$x_\alpha \cup y_\alpha \subseteq Z \equiv Z' \equiv z_\beta \ .$$

Thus $C(x_\alpha), C(y_\alpha), C(Z) \in B_\nu$, so $x, y, z \in B_\nu$.

Theorem 4.5. *Let $\mathcal{F}$ be a consistent complete compact family, then the logic $(L(\mathcal{F}), \leq, \perp)$ generated by $\mathcal{F}$ is an o.m.p. Further, for any $\gamma \in B$, $(F_\gamma)_c$ is an orthogonal subset of $L(\mathcal{F})\setminus\{0\}$, and is a frame of $L(\mathcal{F}) \Leftrightarrow$ there is no element of $\bigcup \mathcal{F}$ which is aloof from F_γ.*

Proof. Using the fact that $\mathcal{L}(\mathcal{F})$ is a logical structure (proved above), $L(\mathcal{F})$ may be shown to be an orthoposet as in the proof of [1, Theorem (1.1)]. We will here prove only that $L(\mathcal{F})$ is orthomodular. Let $x, y \in L(\mathcal{F})$ be orthogonal, and let $x \vee y = 1$. Thus $x \leq_\beta y^\perp$ for some $\beta \in B$. $x \vee_\beta y$ is an upper bound in $L(\mathcal{F})$ to $\{x, y\}$, so $x \vee_\beta y = 1$. B_β is orthomodular (since Boolean), so $x = y^\perp$. Thus $L(\mathcal{F})$ is orthomodular, and the rest of the theorem follows immediately from Theorem 3.3.

In order to obtain further results about the frames of our generated o.m.p., and about the properties of those posets, we will introduce further conditions in addition to the three already introduced. A family $\mathcal{F}$ is *compatible* if, for any $\gamma, \lambda \in B$, if $|F_\gamma \setminus F_\lambda| = 1$ then $|F_\lambda \setminus F_\gamma| \geq 2$.

Lemma 4.6. *Let $\mathcal{F}$ be consistent complete and compatible, then for any $x, y \in \bigcup \mathcal{F}$, if $x_c = y_c$, then $x = y$.*

Proof. By Lemmas 2.1, 2.3(ii) and 2.9(ii).

Let $\mathcal{F}$ be a family, then an (ascending) chain $(X_\delta)_{\delta \in \Delta}$ of subsets of $\bigcup \mathcal{F}$ is an *F^*-chain* if $X_\delta \in F^*$ for all $\delta \in \Delta$. $\mathcal{F}$ is *comprehensive* if for any F^*-chain $(X_\delta)_{\delta \in \Delta}$, $\bigcup\{X_\delta : \delta \in \Delta\} \in F^*$. This condition is introduced at this stage only to take part in a subsiduary theorem, and will appear in its main role in Section 7. It is clear that a finite family is comprehensive.

Theorem 4.7. *Let $\mathcal{F}$ be a consistent complete family such that for any $\beta, \lambda, \mu \in B$, if $F_\lambda \cap F_\mu \neq \emptyset$ then*

$$(1) \qquad F_\beta \cap F_\lambda \subseteq F_\mu \quad \text{or} \quad F_\beta \cap F_\mu \subseteq F_\lambda .$$

Then:

(i) *$\mathcal{F}$ is compact and the logic $(L(\mathcal{F}), \leq, \perp)$ generated by $\mathcal{F}$ is an o.m.p.*

(ii) *If $\mathcal{F}$ is compatible and comprehensive, then there is an o.m.p. P such that for any $\beta \in B$, F_β is a frame of P.*

Proof. (i) Let $\mathcal{F}$ be as stated, and let $Z[\beta_{1\to k}]$ be defined ($k \in E$). By Lemma 2.9(ii) there are $\delta_1, \delta_2 \in B$ such that $Z[\beta_{1\to k}] = Z[\delta_1, \delta_2]$. Let $\alpha \in B$, then to show that $\mathcal{F}$ is compact it is sufficient to show that for some $\nu_1, \nu_2 \in B$, $F_\alpha \cap Z' \subseteq Z[\nu_1, \nu_2]$, where Z' is defined as $Z \cup Z[\delta_1, \delta_2]$.

If $F_{\delta_1} \cap F_{\delta_2} = \emptyset$, then $F_{\delta_1} \setminus Z = \emptyset$, so $F_\alpha \cap Z' \subseteq Z[\delta_1, \alpha]$. Suppose $F_{\delta_1} \cap F_{\delta_2} \neq \emptyset$, then by (1), $F_\alpha \cap F_{\delta_1} \subseteq F_{\delta_2}$ or $F_\alpha \cap F_{\delta_2} \subseteq F_{\delta_1}$. Define $Z_0 = (F_{\delta_1} \cap F_{\delta_2}) \setminus (F_{\delta_1} \setminus Z)$, then

$$F_\alpha \cap Z' = (F_\alpha \cap F_{\delta_1} \setminus F_{\delta_2}) \cup (F_\alpha \cap F_{\delta_2} \setminus F_{\delta_1}) \cup (F_\alpha \cap Z_0) .$$

Thus

$$F_\alpha \cap Z' = (F_\alpha \cap F_{\delta_2} \setminus F_{\delta_1}) \cup (F_\alpha \cap Z_0) \subseteq Z[\delta_1, \delta_2]$$

or

$$F_\alpha \cap Z' = (F_\alpha \cap F_{\delta_1} \setminus F_{\delta_2}) \cup (F_\alpha \cap Z_0) = Z = Z[\delta_1, \delta_1] .$$

Thus $\mathcal{F}$ is compact, so by Theorem 4.5, $(L(\mathcal{F}), \leq, \perp)$ is an o.m.p.

(ii) Suppose $\mathcal{F}$ is comprehensive. Suppose that $x \in \bigcup \mathcal{F}$ is aloof from F_β. Let $\mathcal{S}$ denote the set of all $X \subseteq F_\beta$ such that $\{x\} \cup X \in F^*$. Using the comprehension of $\mathcal{F}$ and Zorn's lemma, we find that $\mathcal{S}$ has a maximal element (wrt $\subseteq$). Detailed considerations, which we omit, involving the application of (1) lead to a contradiction, showing that no element of $\bigcup \mathcal{F}$ is aloof from F_β. By Theorem 4.5, $(F_\beta)_c$ is a frame of $L(\mathcal{F})$.

Now suppose that $\mathcal{F}$ is compatible. By Lemma 4.6, to each element y_c in $(\bigcup \mathcal{F})_c$ there corresponds a unique element of $\bigcup \mathcal{F}$, namely y. Consequently, let P be the o.m.p. obtained from the o.m.p. $L(\mathcal{F})$ by replacing y_c by y for all $y \in \bigcup \mathcal{F}$. Then by the previous paragraph, for any $\beta \in B$, F_β is a frame of P.

5. The universality of the construction

We have shown (Theorem 4.5) how an o.m.p. may be constructed from any consistent complete compact family. In this section we will show that all o.m.p. may be so constructed, more specifically, that given any o.m.p. P there is a consistent complete compact family $\mathcal{F}$ such that the o.m.p. $L(\mathcal{F})$ is isomorphic to P.

Recall that a frame F of a poset P is *complete* if $\bigvee X$ exists in P for all $X \subseteq F$. Until Theorem 5.9 let $(P, \leq, \perp)$ be an arbitrary o.m.p., and let $\mathcal{F}$ be the family of complete frames of P. The following three lemmas are easily proved. The proof of Lemma 5.2 is a simple modification of the proof of [1, Lemma (3.2)], and Lemma 5.3 follows from Lemma 5.2(i) using induction.

Lemma 5.1. *Let F_1 and F_2 be complete frames of P (that is, in $\mathcal{F}$). Let $X \subseteq F_1$ and $Y \subseteq F_2$ such that $X \cap Y = \emptyset$ and $X \cup Y$ is a frame of P, then $X \cup Y$ is in $\mathcal{F}$.*

Lemma 5.2. *Let X, $Y \subseteq F_\beta$, then*
(i) $\bigvee(F_\beta \setminus X) = (\bigvee X)^\perp$, *and*
(ii) $X \subseteq Y \Leftrightarrow \bigvee X \leq \bigvee Y$.

Lemma 5.3. *Let $X \in F^*$ and let $(\beta_i) \in B^n$ $(n \in \mathbf{N})$ such that $X[\beta_{1\to n}]$ is defined, then*

$$\bigvee(X[\beta_{1\to n}]) = \begin{cases} \bigvee X & \text{if } n \in E, \\ (\bigvee X)^\perp & \text{otherwise}. \end{cases}$$

Lemma 5.4. *$\mathcal{F}$ is consistent and complete.*

Proof. The consistency of $\mathcal{F}$ follows from the fact that $Y \subseteq P\setminus\{0\}$ for any $Y \in F^*$, using Lemma 5.3. Let $Y \in F^*$, and define $F = Y \cup Y[\mu_{1\to m}, \beta]$ (supposing $Y[\mu_{1\to m}] \subseteq F_\beta$ $(m \in E)$), then F is orthogonal. From Lemma 5.3, we have $\bigvee F = 1$, so by Lemma 1.1, F is a frame of P, and by Lemma 5.1, F is a complete frame.

Lemma 5.5. *Let $X \subseteq F_\alpha$ and $Y \subseteq F_\beta$ such that $\bigvee X = \bigvee Y$, then there are $\delta_1, \delta_2 \in B$ such that $Y = X[\delta_1, \delta_2]$.*

Proof. $\bigvee X = \bigvee Y$, so by Lemma 1.3 there is a complete frame F of P such that $(\bigvee X)^\perp \in B(F)$. Hence there is $Z \in F^*$ such that $(\bigvee X)^\perp = \bigvee Z$. Define $F_{\delta_1} = X \cup Z$ and $F_{\delta_2} = Y \cup Z$, then by Lemmas 1.1 and 5.1, $F_{\delta_1}, F_{\delta_2} \in \mathcal{F}$. $X \cap Z = \emptyset = Y \cap Z$, so $Y = X[\delta_1, \delta_2]$.

Lemma 5.6. *Let $X \subseteq F_\gamma$ and $Z \subseteq F_\beta$ such that $\bigvee X \leq \bigvee Z$, then there are $\nu_1, \nu_2 \in B$ such that $X \subseteq Z[\nu_1, \nu_2]$.*

Proof. By Lemma 1.3 there is $\lambda \in B$ and $Y, Y' \subseteq F_\lambda$ such that $\bigvee Y = \bigvee X$ and $\bigvee Y' = \bigvee Z$. By Lemma 5.5 there are $\delta_1, \delta_2, \delta_3, \delta_4 \in B$ such that $Y = X[\delta_1, \delta_2]$ and $Y' = Z[\delta_3, \delta_4]$. $\bigvee Y \leq \bigvee Y'$, so by Lemma 5.2(ii), $Y \subseteq Y'$. By Lemma 5.4, $\mathcal{F}$ is consistent and complete, so by Lemma 2.8 there is $Z' \in F^*$ such that

$$X \subseteq Z' \equiv Z[\delta_3, \delta_4] \equiv Z .$$

By Lemma 2.9(ii) there are $\nu_1, \nu_2 \in B$ such that $Z' = Z[\nu_1, \nu_2]$, so $X \subseteq Z[\nu_1, \nu_2]$.

Lemma 5.7. *$\mathcal{F}$ is compact.*

Proof. Let $Z \in F^*$ such that $Z[\beta_{1\to k}]$ is defined $(k \in E)$, and let $\alpha \in B$. By Lemma 5.3, $\bigvee Z = \bigvee Z[\beta_{1\to k}]$, so $\bigvee(F_\alpha \cap (Z \cup Z[\beta_{1\to k}])) \leq \bigvee Z$. The result now follows from Lemma 5.6.

By Theorem 4.5, $(L(\mathcal{F}), \leq, \perp)$ is an o.m.p. We now have only to show that $L(\mathcal{F})$ is isomorphic to P, which we do for the following sense of "isomorphic": Let $(S_1, \leq_1, N_1)$ and $(S_2, \leq_2, N_2)$ be orthocomplemented posets, then S_1 and S_2 are *isomorphic* if there is a bijection (a one-one onto function) $f: S_1 \to S_2$ such that

(i) $f(N_1 x) = N_2(f(x))$, and

(ii) $x \leq_1 y \Leftrightarrow f(x) \leq_2 f(y)$.

It is easily shown that if S_1 and S_2 are isomorphic under f, then for any $X \subseteq S_1$ such that $\mathbf{V}X$ exists in S_1, $f(\mathbf{V}X) = \mathbf{V}f(X)$, where $f(X) = \{f(x): x \in X\}$.

Theorem 5.8. *$L(\mathcal{F})$ and P are isomorphic.*

Proof. For $x \in L(\mathcal{F})$, define $f(x) = \mathbf{V}X$ for any $X \in x$ (the l.u.b. exists by the definition of $\mathcal{F}$). For any $X, X' \in x$ there is $(\beta_i) \in B^n$ $(n \in E)$ such that $X' = X[\beta_{1\to n}]$, so by Lemma 5.3, $\mathbf{V}X' = \mathbf{V}X$. Thus $f: L(\mathcal{F}) \to P$ is well-defined.

By Lemma 5.5, f is an injection. Let $x \in P$, then by Lemma 1.3 there is $\mu \in B$ and $X \subseteq F_\mu$ such that $x = \mathbf{V}X$. $f(C(X)) = \mathbf{V}X = x$, so f is a surjection, and so a bijection.

Let $x \in L(\mathcal{F})$, $X \in x$ and $\beta \in B$ such that $X \subseteq F_\beta$. Then

$$f(x^\perp) = f(C(F_\beta \setminus X)) = \mathbf{V}(F_\beta \setminus X) = (\mathbf{V}X)^\perp = (f(x))^\perp .$$

Let $x, y \in L(\mathcal{F})$ such that $x \leq y$. Then $x \leq_\beta y$ for some $\beta \in B$, so $x_\beta \subseteq y_\beta$. Thus $\mathbf{V}x_\beta \leq \mathbf{V}y_\beta$, so $f(x) \leq f(y)$. Conversely, suppose $f(x) \leq f(y)$, then $\mathbf{V}X \leq \mathbf{V}Y$ for some $X \in x$ and $Y \in y$. By Lemma 5.6 there are $\nu_1, \nu_2 \in B$ such that $X \subseteq Y[\nu_1, \nu_2]$, so

$$x = C(X) \leq C(Y[\nu_1, \nu_2]) = C(Y) = y .$$

Thus $L(\mathcal{F})$ and P are isomorphic.

Theorem 5.9. (i) *Every o.m.p. is such that its family of complete frames is consistent, complete and compact.*

(ii) *From any given consistent complete compact family there can be constructed an o.m.p.*

(iii) *The o.m.p. constructed from the family of complete frames of an o.m.p. P is isomorphic to P.*

Proof. By Theorems 4.5 and 5.8.

The $\mathcal{F}$ from which a given o.m.p. P may be constructed may not be unique. A problem which deserves further investigation is whether, for a given P, there is always some $\mathcal{F}$ from which (an isomorphic copy of) P can be constructed which is in some sense unique among such $\mathcal{F}$'s. For example, is there always such an $\mathcal{F}$ which is minimal in the sense of being properly contained in any other such $\mathcal{F}$?

6. Families of frames

Let $\mathcal{F}$ be a consistent complete compact family, then by Theorem 4.5, for any $\beta \in B$, $(F_\beta)_c$ is an orthogonal subset of the o.m.p. $L(\mathcal{F})$ generated by $\mathcal{F}$. But $(F_\beta)_c$, although contained in $L(\mathcal{F})\backslash\{0\}$, may not be a frame of $L(\mathcal{F})$. The question arises as to what further condition $\mathcal{F}$ must satisfy so that $(F_\beta)_c$ is a frame of $L(\mathcal{F})$ for all $\beta \in B$. By Theorem 4.5, we have only to find a condition which eliminates the possibility of $\bigcup\mathcal{F}$ containing an aloof element. The condition defined below accomplishes this, and also ensures that the $(F_\beta)_c$ are *complete* frames of $L(\mathcal{F})$.

A family $\mathcal{F}$ is *continuous* if for any $\beta \in B$, any (ascending) chain $(X_\delta)_{\delta\in\Delta}$ in F_β, and any $Y \in F^*$, if for any $\delta \in \Delta$ there are $\gamma_1, \gamma_2 \in B$ such that $X_\delta \subseteq Y[\gamma_1, \gamma_2]$, then there are $\gamma_1, \gamma_2 \in B$ such that $\bigcup\{X_\delta : \delta \in \Delta\} \subseteq Y[\gamma_1, \gamma_2]$. Clearly, if the sets in $\mathcal{F}$ are all finite, then $\mathcal{F}$ is continuous. For $X \subseteq \bigcup\mathcal{F}$, define $X_c = \{x_c : x \in X\}$, where x_c has earlier been defined as $C(\{x\})$. Clearly, $X_c \subseteq L(\mathcal{F})$.

Lemma 6.1. *Let $\mathcal{F}$ be a consistent complete compact continuous family. Let $X \in F^*$ and let $u \in L(\mathcal{F})$ be such that u is an upper bound in $L(\mathcal{F})$ to X_c, then there is $Y \in u$ such that $X \subseteq Y$.*

Proof. Let $u \in L(\mathcal{F})$ be an upper bound in $L(\mathcal{F})$ to X_c, and define $\mathcal{S}$ as the set of all $S \subseteq X$ such that there is $Y \in u$ such that $S \subseteq Y$. For any $x \in X$, $\{x\} \in \mathcal{S}$, so $\mathcal{S} \neq \emptyset$. Using Zorn's lemma and the continuity of $\mathcal{F}$ (and Lemma 2.9(ii)), $\mathcal{S}$ may be shown to have a maximal element (wrt $\subseteq$) S'. $S' \subseteq X$, and the assumption of proper inclusion leads to a contradiction of the maximality of S' (with the aid of Lemma 4.1). Since $X = S' \in \mathcal{S}$, there is $Y \in u$ such that $X \subseteq Y$.

Lemma 6.2. *A consistent complete compact continuous family* $\mathcal{F}$ *is such that* $\bigcup\mathcal{F}$ *contains no aloof element.*

Proof. Suppose $\mathcal{F}$ contains an aloof element x. From the definition of aloofness it follows that there is $\beta \in B$ such that $x \notin F_\beta$ and $x_c^\perp$ is an upper bound in $L(\mathcal{F})$ to $(F_\beta)_c$. By Lemma 6.1 there is $Y \in x_c^\perp$ such that $F_\beta \subseteq Y$. Let $\delta \in B$ be such that $x \in F_\delta$, then

$$x \in F_\delta \equiv F_\beta \subseteq Y \in x_c^\perp$$

so

$$x_c \leq C(F_\delta) = C(F_\beta) \leq C(Y) = x_c^\perp$$

so $x_c = 0$, which is impossible. Thus $\bigcup\mathcal{F}$ contains no aloof element.

Lemma 6.3. *In such a family* $\mathcal{F}$, $\bigvee(X_c) = C(X)$ *for any* $X \in F^*$.

Proof. Let $X \in F^*$, then clearly $C(X)$ is an upper bound in $L(\mathcal{F})$ to X_c. Let u be any such upper bound, then by Lemma 6.1 there is $Y \in u$ such that $X \subseteq Y$. $C(X) \leq C(Y) = u$, so $C(X)$ is the least upper bound in $L(\mathcal{F})$ to X_c.

So far we have introduced six conditions:
(I) consistency (Section 2),
(II) completeness (Section 2),
(III) compactness (Section 4),
(IV) compatibility (Section 4),
(V) continuity (Section 6),
(VI) comprehension (Section 4).
We will later reintroduce comprehension (Section 7) and introduce:
(IVS) strong compatibility (Section 8).
From now on in the statement of lemmas and theorems we will refer to these conditions by the corresponding Roman numeral.

Lemma 6.4. *Let* $\mathcal{F}$ *be a family satisfying* (I)–(III) *and* (V), *then for any* $\beta \in B$, $(F_\beta)_c$ *is a complete frame of* $L(\mathcal{F})$.

Proof. By Lemma 6.2, $\bigcup\mathcal{F}$ contains no aloof element, so by Theorem

3.3, $(F_\beta)_c$ is a frame of $L(\mathcal{F})$. Let $X \subseteq (F_\beta)_c$, and define $X_{c*} = \{x \in F_\beta: x_c \in X\}$. Now $X_{c*} \in F^*$, so by Lemma 6.3, $\mathbf{V}((X_{c*})_c) = C(X_{c*})$. It is easily shown than $x \in (X_{c*})_c \Leftrightarrow x \in X$, so $(X_{c*})_c = X$, so $\mathbf{V}X = C(X_{c*})$. Hence $(F_\beta)_c$ is a complete frame of $L(\mathcal{F})$.

Theorem 6.5. *Let $\mathcal{F}$ be a family satisfying* (I)–(V), *then there is an o.m.p. P such that for any $\beta \in B$, F_β is a complete frame of P.*

Proof. By Theorem 4.5 and Lemma 6.4, for any $\beta \in B$, $(F_\beta)_c$ is a complete frame of the o.m.p. $L(\mathcal{F})$. $\mathcal{F}$ is compatible, so as in the last paragraph of the proof of Theorem 4.7, let P be the o.m.p. obtained from $L(\mathcal{F})$ by replacing y_c by y for all $y \in \mathbf{U}\mathcal{F}$, then for any $\beta \in B$, F_β is a complete frame of the o.m.p. P.

Lemma 6.6. *Let $(P, \leq, \perp)$ be an o.m.p. and let $\mathcal{F}$ be the family of complete frames of P, then $\mathcal{F}$ satisfies* (I)–(V).

Proof. By Theorem 5.8, $\mathcal{F}$ satisfies (I)–(III). Using Lemma 5.3, we find that $\mathcal{F}$ satisfies (IV). To prove the continuity of $\mathcal{F}$, consider $\beta \in B$, $Y \in F^*$, and a chain $(X_\delta)_{\delta \in \Delta}$ in F_β such that for all $\delta \in \Delta$ there are $\gamma_1, \gamma_2 \in B$ such that $X_\delta \subseteq Y[\gamma_1, \gamma_2]$. By Lemma 5.3, $\mathbf{V}X_\delta \leq \mathbf{V}Y$ for all $\delta \in \Delta$. F_β is complete and $\mathbf{U}\{X_\delta : \delta \in \Delta\} \subseteq F_\beta$, so $\mathbf{V}(\mathbf{U}\{X_\delta\})$ exists in P. Clearly, $\mathbf{V}(\mathbf{U}\{X_\delta\}) = \mathbf{V}\{\mathbf{V}X_\delta\}$. Y is an upper bound in P to $\{\mathbf{V}X_\delta\}$, so $\mathbf{V}(\mathbf{U}\{X_\delta\}) \leq \mathbf{V}Y$. An application of Lemma 5.6 completes the proof.

Theorem 6.7. (i) *Every o.m.p. is such that its family of complete frames satisfies* (I)–(V).

(ii) *From any given family $\mathcal{F}$ satisfying* (I)–(V) *there can be constructed an o.m.p. whose family of complete frames includes $\mathcal{F}$.*

(iii) *The o.m.p. constructed from the family of complete frames of an o.m.p. P is isomorphic to P.*

Proof. By Theorems 5.9 and 6.5, and by Lemma 6.6.

7. Completely orthomodular posets

In the remainder of the paper we will obtain three further theorems of the form of Theorem 6.7, namely, results for (i) completely o.m.p., (ii) "A-logics", and (iii) "weakly atomic" completely o.m.p. For the sake of brevity we will (as before) frequently omit proofs or only briefly describe them.

To obtain *complete* orthomodularity in our constructed o.m.p. we require that $\mathcal{F}$ be comprehensive, which notion was defined just prior to Theorem 4.7. It will be recalled that a chain $(X_\delta)_{\delta\in\Delta}$ in $\bigcup\mathcal{F}$ is an *F^*-chain* if $X_\delta \in F^*$ for all $\delta\in\Delta$, and that $\mathcal{F}$ is *comprehensive* if for any F^*-chain $\mathcal{C}$, $\bigcup\mathcal{C}\in F^*$. The index sets Δ are ordinals, and so are well-ordered.

Lemma 7.1. *Let $\mathcal{F}$ be a family satisfying* (I)–(III) *and* (V)–(VI), *and let X be an orthogonal subset of* $(\bigcup\mathcal{C})_c \subseteq L(\mathcal{F})$, *then there is $Z\in F^*$ such that $Z_c = X$.*

Proof. The proof, which in full is lengthy, considers the set C of all F^*-chains $\mathcal{C}$ such that $(\bigcup\mathcal{C})_c \subseteq X$, and the partial ordering $\leq$ on C holding between two F^*-chains $(X_\delta)_{\delta\in\Delta}$ and $(X'_\delta)_{\delta\in\Delta'}$ when $\Delta\leq\Delta'$ and for all $\delta\in\Delta$, $X_\delta\subseteq X'_\delta$. Zorn's lemma implies the existence of a maximal element $\mathcal{C}'$ in $(C,\leq)$. $(\bigcup\mathcal{C}')_c\subseteq X$, and the assumption that the inclusion is proper leads to a contradiction of the maximality of $\mathcal{C}'$ (using Lemma 6.1), so $(\bigcup\mathcal{C}')_c = X$. $\mathcal{F}$ is comprehensive and $\mathcal{C}'$ is an F^*-chain, so $\bigcup\mathcal{C}'\in F^*$. Hence there is $Z\in F^*$ such that $Z_c = X$.

Let P be an orthocomplemented poset, then $Y\subseteq P$ is a *section* of P if every nonzero element of P dominates some element of Y. A family $\mathcal{F}$ of frames of P is *sectionally complete* if $\bigcup\mathcal{F}$ is a section of P and $\mathcal{F}$ is the family of all maximal orthogonal subsets of $\bigcup\mathcal{F}$. The following lemma is elementary.

Lemma 7.2. *Let $\mathcal{F}$ be a consistent complete family, then $(\bigcup\mathcal{F})_c$ is a section of $(L(\mathcal{F}),\leq,\perp)$.*

Theorem 7.3. *Let $\mathcal{F}$ be a family satisfying* (I)–(III) *and* (V)–(VI), *then the logic $(L(\mathcal{F}),\leq,\perp)$ generated by $\mathcal{F}$ is a completely o.m.p., and $\mathcal{F}_c$ is* *a sectionally complete family of frames of* $L(\mathcal{F})$, *where* $\mathcal{F}_c$ *is defined as* $\{(F_\beta)_c : \beta \in B\}$.

Proof. $L(\mathcal{F})$ is an o.m.p. by Theorem 4.5. Let Y be an orthogonal subset of $L(\mathcal{F})$, and for $y \in Y$, let $[y]$ be any maximal orthogonal subset of elements of $(\bigcup\mathcal{F})_c$ dominated by y. By Lemmas 7.2 and 1.2, $y = \bigvee[y]$. Define $X = \bigcup\{[y] : y \in Y\}$, then X is an orthogonal subset of $(\bigcup\mathcal{F})_c$, so by Lemmas 7.1 and 6.4, $\bigvee X$ exists in $L(\mathcal{F})$. $\bigvee X = \bigvee Y$, so $L(\mathcal{F})$ is completely orthomodular.

By Lemma 6.4, $\mathcal{F}_c$ is a family of frames of $L(\mathcal{F})$. $\bigcup(\mathcal{F}_c) = (\bigcup\mathcal{F})_c$, so by Lemma 7.2, $\bigcup(\mathcal{F}_c)$ is a section of $L(\mathcal{F})$. To show that $\mathcal{F}_c$ is sectionally complete we have only to show that $\mathcal{F}_c$ is the family of all maximal orthogonal subsets of $(\bigcup\mathcal{F})_c$. Let $\beta \in B$, then by Lemma 6.4, $(F_\beta)_c$ is a frame of $L(\mathcal{F})$, and so is a m.o.s. of $(\bigcup\mathcal{F})_c$. Conversely, suppose that W is a m.o.s. of $(\bigcup\mathcal{F})_c$, then by Lemma 7.1 there is $Z \in F^*$ such that $Z_c = W$. For some $\beta \in B$, $Z \subseteq F_\beta$, so $W = Z_c \subseteq (F_\beta)_c$. $(F_\beta)_c$ is an orthogonal subset of $(\bigcup\mathcal{F})_c$ and W is maximal, so $W = (F_\beta)_c$. Thus $W \in \mathcal{F}_c$, which completes the proof.

Theorem 7.4. (i) *Every completely o.m.p. is such that its family of frames is sectionally complete and satisfies* (I)–(VI).

(ii) *From any given family* $\mathcal{F}$ *satisfying* (I)–(VI) *there can be construc ted a completely o.m.p. which has* $\mathcal{F}$ *as a sectionally complete family of frames.*

(iii) *The completely o.m.p. constructed from the family of frames of a completely o.m.p. P is isomorphic to P.*

Proof. Let $(P, \leq, \perp)$ be a completely o.m.p., and let $\mathcal{F}$ be the family of frames of P, necessarily complete frames. By Theorem 6.7, $\mathcal{F}$ satisfies (I)–(V), and $L(\mathcal{F})$ and P are isomorphic. $\mathcal{F}$ is the family of all frames of P, and so of all maximal orthogonal subsets of $\bigcup\mathcal{F}$, so $\mathcal{F}$ is sectionally complete. A demonstration of the comprehension of $\mathcal{F}$ will complete the proof of (i) and (iii). Let $\mathcal{C}$ be an F^*-chain, then $\bigcup\mathcal{C}$ is the union of a chain of orthogonal subsets of $P\setminus\{0\}$, and so is an orthogonal subset of $P\setminus\{0\}$. By Zorn's lemma, $\bigcup\mathcal{C}$ can be extended to a frame of P. $\mathcal{F}$ contains all frames of P, so $\bigcup\mathcal{C} \in F^*$. Thus $\mathcal{F}$ is comprehensive.

To prove (ii) let $\mathcal{F}$ be a family satisfying (I)–(VI), then by Theorem

7.3, $(L(\mathcal{F}'), \leq, \perp)$ is a completely o.m.p. and $\mathcal{F}_c = \{(F_\beta)_c : \beta \in B\}$ is a sectionally complete family of frames of $L(\mathcal{F})$. As in the proof of Theorem 6.5, let P be the completely o.m.p. obtained from $L(\mathcal{F})$ by replacing y_c by y for all $y \in \bigcup \mathcal{F}$, then $\mathcal{F}$ is a sectionally complete family of frames of P.

8. A-logics

For a poset $(P, \leq)$ with 0-element, $x \in P\setminus\{0\}$ is an *atom* if $y \leq x$ implies $y = x$ or $y = 0$. Let $\mathcal{F} = \{\{a, b\}, \{b, c, d\}\}$, then $\mathcal{F}$ satisfies (I)–(VI), so $(L(\mathcal{F}), \leq, \perp)$ is an o.m.p., namely, (an isomorphic copy of) the Boolean lattice of all subsets of $\{b, c, d\}$. a_c is not an atom of $L(\mathcal{F})$ since $c_c < a_c$. Thus it is not in general true that if a family satisfies (I)–(VI), then $(\bigcup \mathcal{F})_c$ is (or is contained in) the set of atoms of $L(\mathcal{F})$. In fact, there exists a family $\mathcal{F}'$ satisfying (I)–(V) such that *no* element of $(\bigcup \mathcal{F})_c$ is an atom of $L(\mathcal{F})$. This follows, by Lemma 6.6, from the fact that there exist Boolean lattices which do not possess atoms. In this section we will introduce a condition which (together with consistency and completeness) ensures that $(\bigcup \mathcal{F})_c$ is the set of atoms of $L(\mathcal{F})$.

Let $(P, \leq, \perp)$ be an orthoposet. A frame of P is *atomic* if it consists entirely of atoms of P. P is an A-*logic* if for any $x, y \in P$ such that $x \leq y$, there is a complete atomic frame F of P such that $x, y \in B(F)$, where $B(F)$ was earlier defined as $\{\bigvee X : X \subseteq F\}$. It is clear that $B(F)$ is a Boolean lattice isomorphic to that of all subsets of F. P is *weakly atomic* if every nonzero element of P dominates some atom of P. Clearly, an A-logic is weakly atomic, although a weakly atomic orthoposet need not be an A-logic. The definition of an A-logic should be compared with Lemma 1.3, from which it follows that an A-logic is an o.m.p. (necessarily weakly atomic). To see that a weakly atomic o.m.p. need not be an A-logic we have only to consider the Boolean lattice B of all subsets of $\mathbf{N}$ which are finite or have finite complement in $\mathbf{N}$. B is weakly atomic, possesses one atomic frame, and no complete atomic frame, so B is not an A-logic.

Lemma 8.1. *Every weakly atomic completely o.m.p. is an A-logic.*

Proof. Using Lemma 1.2 twice and Zorn's lemma three times.

Later in this section an example will be given of an A-logic which is not completely orthomodular. Thus the class of weakly atomic completely o.m.p. is included in the class of A-logics, which is included in the class of weakly atomic o.m.p., and in both cases the inclusion is proper.

A family $\mathcal{F}$ is *strongly compatible* if $|F_\beta \setminus F_\gamma| \neq 1$ for all $\beta, \gamma \in B$. It holds trivially that a strongly compatible family is compatible. We denote strong compatibility by (IVS), so that any family satisfying (IVS) satisfies (IV) also.

Theorem 8.2. (i) *Every A-logic is such that its family of complete atomic frames satisfies* (I)–(III), (IVS) *and* (V).

(ii) *From any given family $\mathcal{F}$ satisfying* (I)–(III), (IVS) *and* (V) *there can be constructed an A-logic whose family of complete atomic frames includes $\mathcal{F}$.*

(iii) *The A-logic constructed from the family of complete atomic frames of an A-logic P is isomorphic to P.*

Proof. (i) Let $(P, \leq, \perp)$ be an A-logic, then by Lemma 1.3, P is an o.m.p. Let $\mathcal{F}$ be the family of complete atomic frames of P. To show that $\mathcal{F}$ is consistent, complete, compact, and continuous, it suffices to go through the proofs of Lemmas 5.2 to 5.7, and that of Lemma 6.6, to ascertain that they remain valid when P is an A-logic (not just an o.m.p.) and $\mathcal{F}$ is the family of complete *atomic* frames of P. Since P is an A-logic, F in Lemma 5.5 and F_λ in Lemma 5.6 may be chosen so as to be atomic frames. These are the only modifications necessary, so $\mathcal{F}$ satisfies (I)–(III) and (V). Suppose now that $\mathcal{F}$ is not strongly compatible, then there are $\beta, \gamma \in B$ and an atom a of P such that $F_\beta \setminus F_\gamma = \{a\}$. There is an atom $b \in F_\gamma \setminus F_\beta$, and

$$\begin{aligned} b \leq \mathbf{V}(F_\gamma \setminus F_\beta) &= \mathbf{V}((F_\beta \setminus F_\gamma)[\beta, \gamma]) \\ &= \mathbf{V}(F_\beta \setminus F_\gamma) \quad \text{by Lemma 5.3}, \\ &= a\,. \end{aligned}$$

Thus $b \leq a$, and a and b are atoms, so $a = b$, so $b \notin F_\gamma$, a contradiction. Hence $\mathcal{F}$ satisfies (IVS) also.

(ii) Let $\mathcal{F}$ be as stated, then by Theorem 4.5, $L(\mathcal{F})$ is orthomodular, and so is an orthoposet. By Lemmas 3.4 and 6.4, for any $\beta \in B$, $(F_\beta)_c$ is a complete atomic frame of $L(\mathcal{F})$. Let $x, y \in L(\mathcal{F})$ such that $x \leq y$, then $x_\beta \subseteq y_\beta$ for some $\beta \in B$. By Lemma 6.3, $x = C(x_\beta) = \mathbf{V}((x_\beta)_c)$ and $y = C(y_\beta) = \mathbf{V}((y_\beta)_c)$, so $x, y \in B((F_\beta)_c)$. Since $(F_\beta)_c$ is a complete atomic frame of $L(\mathcal{F})$, it follows that $L(\mathcal{F})$ is an A-logic. $\mathcal{F}$ is compatible, so as in the proof of Theorem 6.5 there is an A-logic P such that for any $\beta \in B$, F_β is a complete atomic frame of P.

(iii) As in the proof of (i) it suffices to check that the proof of Theorem 5.8 remains valid when P is an A-logic and $\mathcal{F}$ is the family of complete atomic frames of P (remembering that Lemmas 5.2 to 5.7 are so valid). Since F_μ may be chosen so as to be an atomic frame, the proof does in fact remain valid.

An unsolved problem: Let $L(\mathcal{F})$ be the A-logic generated by a family satisfying (I)–(III), (IVS) and (V), then is $\{(F_\beta)_c : \beta \in B\}$ identical to, and not just contained in (as proved above), the family of complete atomic frames of $L(\mathcal{F})$? This question has a negative answer if we omit "complete", as can be found by considering the family $\mathcal{F} = \{F_n : n \in \mathbf{N}\}$, where F_n is defined as $\{a_1, a_2, ..., a_n, b_n, c_n\}$. It can be shown that $\mathcal{F}$ satisfies (I)–(III), (IVS) and (V), so by Theorem 8.2, $L(\mathcal{F})$ is an A-logic. Define $A = \{a_2, a_4, a_6, ...\}$, then A_c is an orthogonal set of atoms of $L(\mathcal{F})$, and so can be extended to an atomic frame F of $L(\mathcal{F})$, and it is clear that $F \notin \mathcal{F}$. It can also be shown that A_c has no l.u.b. in $L(\mathcal{F})$, so F is not a complete frame. Thus, as promised earlier, $L(\mathcal{F})$ is an instance of an A-logic which is not a complete orthoposet, and so is not completely orthomodular.

9. Weakly atomic completely orthomodular posets

Theorem 9.1. (i) *Every weakly atomic completely o.m.p. is such that its family of atomic frames satisfies* (I)–(III), (IVS) *and* (V)–(VI).

(ii) *From any given family $\mathcal{F}$ satisfying these conditions there can be constructed a weakly atomic completely o.m.p. whose family of atomic frames is $\mathcal{F}$.*

(iii) *The weakly atomic completely o.m.p. constructed from the family of atomic frames of a weakly atomic completely o.m.p. P is isomorphic to P.*

Proof. (i) Let $(P, \leq, \perp)$ be a weakly atomic completely o.m.p. and let $\mathcal{F}$ be the family of atomic (necessarily complete) frames of P. By Lemma 8.1, P is an A-logic. By Lemma 8.2(i), $\mathcal{F}$ satisfies (I)–(III), (IVS) and (V). By adding an "atomic" here and there in the demonstration of comprehension in the proof of Theorem 7.4, we find that $\mathcal{F}$ satisfies (VI).

(ii) Let $\mathcal{F}$ be as stated, then by Theorem 7.3, $L(\mathcal{F})$ is a completely o.m.p. and $\mathcal{F}_c$ is a sectionally complete family of frames of $L(\mathcal{F})$. Thus $\mathcal{F}_c$ is the family of all maximal orthogonal subsets of $\bigcup(\mathcal{F}_c)$. By Lemma 3.4, $\bigcup(\mathcal{F}_c) = (\bigcup \mathcal{F})_c$ is the set of atoms of $L(\mathcal{F})$, so $\mathcal{F}_c$ is the family of all atomic frames of $L(\mathcal{F})$. Since $\mathcal{F}_c$ is sectionally complete, $(\bigcup \mathcal{F})_c = \bigcup(\mathcal{F}_c)$ is a section of $L(\mathcal{F})$, so every nonzero element of $L(\mathcal{F})$ dominates some element of $(\bigcup \mathcal{F})_c$ and so dominates some atom of $L(\mathcal{F})$, so $L(\mathcal{F})$ is weakly atomic. Since $\mathcal{F}$ is compatible, we may replace the elements of each $(F_\beta)_c$ by those of F_β (as in the proof of Theorem 6.7) to obtain our result.

(iii) This follows directly from Lemma 8.1 and Theorem 8.2(iii).

In order to demonstrate that a given family $\mathcal{F}$ generates an o.m.p. (respectively, a weakly atomic completely o.m.p.), it is more convenient in practice to show that $\mathcal{F}$ satisfies certain conditions which, although perhaps stronger than (I)–(III) (respectively, (I)–(III), (IVS) and (V) (VI)), are yet easier to demonstrate as holding for $\mathcal{F}$. The two theorems following are steps in this direction. Let $\mathcal{F} = \{F_\beta \colon \beta \in B\}$, then $(\beta_i) \in B^n$ is *irredundant* if $F_{\beta_i} \neq F_{\beta_{i+1}}$ for all $i \in \mathbf{N}$ such that $1 \leq i < n$.

Theorem 9.2. *Let $\mathcal{F}$ be finite and such that*

(i) *for any $X \in F^*$, if $X[\beta_{1\to n}]$ is defined ($n \in E$) and (β_i) is irredundant, then $n \leq 2$; and*

(ii) *if $F_\lambda \cap F_\mu \neq \emptyset$, then $F_\gamma \cap F_\lambda \subseteq F_\mu$ or $F_\gamma \cap F_\mu \subseteq F_\lambda$;*

then the logic $L(\mathcal{F})$ generated by $\mathcal{F}$ is an o.m.p. and each $(F_\beta)_c$ is a frame thereof.

Proof. Condition (i) implies that $\mathcal{F}$ is consistent and complete (this implication not depending on the finiteness of $\mathcal{F}$). Since a finite family is comprehensive, the result follows from (the proof of) Theorem 4.7.

Theorem 9.3. *Let $\mathcal{F}$ be such that*

(i) *if $F_\gamma \neq F_\lambda$, then $|F_\gamma \cap F_\lambda| \leq 1$;*

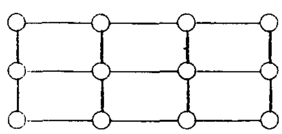
(ii) $|F_\gamma \setminus F_\lambda| \neq 1$; *and*

(iii) *if $F_\lambda \neq F_\mu$ and $|F_\gamma \cap F_\lambda| = 1 = |F_\gamma \cap F_\mu|$, then $F_\lambda \cap F_\mu = \emptyset$;*

then there is a weakly atomic completely o.m.p. whose family of atomic frames is $\mathcal{F}$.

Proof. From (i) and (iii) we may prove condition (ii) of Theorem 9.2, showing that $\mathcal{F}$ is consistent and complete. Compactness is proved using Theorem 4.7(i). (ii) is just strong compatibility, and continuity and comprehension follow from (ii) and (iii) by elementary (but lengthy) argument. The result then follows from Theorem 9.1(ii).

The usefulness of such results is that they permit us to consider complicated o.m.p. by means of far less complicated representations of families (for example, the A-logic generated by the family $\{F_n : n \in \mathbf{N}\}$ defined earlier). As a simpler example, consider the family $\mathcal{F}$ whose seven elements are the colinear sets below:



By Theorem 9.3 there is a weakly atomic completely o.m.p. P whose family of atomic frames is $\mathcal{F}$ ($|P| = 44$). Suppose P admits a normed orthovaluation $p : P \to [0, 1]$, then (as is true of any orthoposet P) $\Sigma\{p(x): x \in F\} = 1$ for any frame F of P. By considering first the horizontal frames of P and then the vertical ones, we conclude that $3 = 4$ (as in [3]). Thus P does not admit a normed orthovaluation. Hence we see how this investigation of the structure of o.m.p. may prove useful in the practical matter of finding counter-examples to false conjectures in the theory of orthomodular posets.

Note added in proof. The results in this paper were all obtained in 1970 during the author's final undergraduate year at Monash University. They constitute the first and the last contribution to mathematics by the author, who, preferring to work in the field of Mahāyāna Buddhist philosophy, invites all interested mathematicians to make whatever use they they can of these results.

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